Highly Efficient, Very Low-Thrust Transfer to Geosynchronous Orbit: Exact and Approximate Solutions

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Very low-thrust minimum-fuel orbit transfers for spacecraft with constant specific-impulse propulsion systems require multiple burns at perigee and apogee to keep gravity losses small. A lengthy iterative computer program has been developed to solve for the optimal thrust direction histories of such transfers, given specific numbers of burns, for fixed-thrust and fixed-acceleration cases. Algebraic approximations for total time and loss are presented that provide quick determination of the best numbers of burns vs trip time. Results for the low Earth orbit to geosynchronous orbit case show the approximations to be good; moreover, for long trip times, loss varies inversely as the square of the acceleration limit for fixed trip time, and loss varies inversely as the square of time for fixed-acceleration limits.

Introduction

A NUMBER of space missions are contemplated that will benefit from or require a low limit on thrust-acceleration during transfer from low Earth orbit to a high orbit. Some of these are large strucures of lightweight, flimsy construction, erected or deployed in low orbit and subsequently moved to their operational orbits, such as geosynchronous. Others are more conventional spacecraft, which can benefit from low thrust by using smaller propulsion systems, or perhaps by combining station-keeping and transfer propulsion systems. In some cases, it may be required to perform the transfer at a limiting acceleration as low as $(1/1000)g$. In all cases it is desirable to minimize the fuel required for transfer by finding the optimal thrust burn times and direction history.

The major difference between a low-thrust maneuver and a high-thrust or impulsive maneuver of equivalent effect is the greater length of the low-thrust burn-arcs. As the thrust is lowered, more acceleration is applied farther from the optimal points—the transfer orbit perigee and apogee points in circle-to-circle transfer—so that the total velocity increment $\Delta V = \int A(t)\,dt$ also grows. The difference between the low-thrust ΔV and the ΔV_{imp} for an equivalent impulsive maneuver is termed gravity loss; it is a measure of the fuel efficiency of a maneuver.

It is possible to reduce gravity loss to an arbitrarily low level, and thereby reduce the total maneuver ΔV nearly to the impulsive value, for any thrust level, provided one is willing to take a large penalty in trip time. Loss is reduced by subdividing the perigee and apogee burns into several intermediate burns with shorter average arc lengths, which increases the proportion of the ΔV applied near the optimum points. Each intermediate burn produces a larger intermediate orbit until the desired final orbit is attained, with each added orbit increasing trip time by roughly the time of its period. An example transfer from low Earth orbit to geosynchronous orbit (LEO-GEO transfer) is sketched in Fig. 1.

For variable thrust propulsion systems, it is best to burn at the maximum allowed thrust, as this will shorten the burn-arcs and so concentrate the acceleration near the optimal points. The best thrust pointing direction can be determined: It differs from both the local horizontal (LOH) and the direction of the velocity. The optimal thrust vector typically points inward of the LOH initially, rotating inward at an average rate less than the initial orbital rate, so that at the end of the burn it points outward of the LOH. There is a varying out-of-plane component of thrust to perform any desired plane change.

Due to the greater contribution to gravity loss of the perigee burns, as well as the greater speeds at perigee, the greatest savings comes from the division of the perigee burns. Lower apogee speeds mean that longer time, higher ΔV burns are possible at apogee than at perigee, for comparable burn-arc lengths and gravity loss. Hence more perigee than apogee burns are required, for the same contribution to the total gravity loss. The optimal numbers of perigee and apogee burns are a tradeoff that can be decided by enforcing a limit on trip time: The best numbers of burns for a transfer are those which give the least loss in a given time.

A large iterative computer program was developed[1] to solve for the optimal burn times and thrust direction history, total time, and loss for transfers of this type, given the initial thrust-to-weight ratio I_{sp}, number of perigee burns N, number of apogee burns M, and initial and final orbit states. The program is exact for the unperturbed problem (spherical Earth, drag-free, two-body problem). The optimal division of burns can be found by searching with this program: By fixing the thrust I_{sp} and time limit and searching over N and M to find the N-M combination that defines the transfer with the least loss and a trip time less than the time limit. This procedure is effective but expensive.

In order to fully develop the solutions to this class of problem without incurring prohibitive computer bills, it was convenient to develop approximate solutions to replace the expensive exact solution. These approximations are described in this paper.

Exact and approximate results are presented for three general cases in the low Earth to geosynchronous orbit transfer problem: 1) constant acceleration, 2) constant thrust, and 3) "discrete throttling" propulsion schemes. For transfers with an absolute limit on acceleration, the best way to burn is to vary the thrust so as to keep acceleration at the limit throughout the burn. Fixed-thrust transfers have their maximum acceleration at the end of the transfer when the spacecraft is lightest; during most of the transfer, acceleration is lower than the limit, incurring more gravity loss. Discrete

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throttling refers to an intermediate scheme whereby the initial thrust is adjusted between burns so that the maximum acceleration during each (constant thrust) burn is the limit acceleration.

The Exact Solution

The exact solution is computed following the optimal orbit transfer theory first developed by Lawden, according to which the optimal thrust direction and burn times are given by the behavior of the adjoint vector λ_v:

$$\text{state vector} = [r, v, (m/m_0)]$$

$$\text{adjoint vector} = [\lambda_r, \lambda_v, \lambda_m]$$

The optimal control is given as

$$\max_{T(t)} \left[H = -\lambda_m \frac{T}{c} + \lambda_r \cdot v + \lambda_v \cdot \left(-\frac{\mu r}{r^3} + \frac{T}{m}\hat{\beta} \right) \right]$$

which implies that

$$T = T_M \quad \text{when} \quad |\lambda_v| > m\lambda_m / c$$

$$= 0 \quad \text{when} \quad |\lambda_v| < m\lambda_m / c$$

$$\hat{\beta} = \lambda_v \, |\lambda_v|$$

The equations of motion in state vector form:

$$\dot{r} = v$$

$$\dot{v} = -(\mu r / r^3) + T\hat{\beta}$$

$$\dot{m}/m_0 = -(T/m_0 c)$$

$$\dot{\lambda}_r = -\Gamma(r) \cdot \lambda_v$$

$$\dot{\lambda}_v = -\lambda_r$$

$$\dot{\lambda}_m = T|\lambda_v|/m^2$$

Here $\Gamma(r)$ is the gravity gradient tensor.

Solution is by the shooting method whereby (for a given desired initial and final forbit, I_{sp}, T/m_0, N, and M, etc.):

1) Initial adjoints and position in the initial orbit are guessed.

2) The state vector is integrated forward, with thrust T on or off, depending on λ_v, as above.

3) Conditions at the final shutoff are compared with the desired final conditions and the initial guesses improved accordingly.

4) The procedure is repeated as necessary.

A more detailed description is available in Ref. 1.

Approximate Solution for Fixed-Acceleration Transfers

The basis of each of our approximate solutions is a formula for gravity loss given by Robbins[2] that provides an estimate of the loss of a low-thrust maneuver based on the ΔV, radius, and turn rate of an impulsive maneuver of equivalent effect. Robbins's formula is

$$\text{Loss} = (\Delta v - \Delta v_{\text{imp}}) = \frac{1}{2}\left(\frac{\mu}{r^3} - \omega_{\text{opt}}^2\right)\int_{t_0}^{t_B} (t - t_c)^2 A(t)\,\mathrm{d}t \tag{1}$$

where t_c is such that

$$\int_{t_0}^{t_B} (t - t_c) A(t)\,\mathrm{d}t = 0$$

and ω_{opt} is the turn rate of the optimal thrust direction. For the constant acceleration case, we have (for a single burn)

$$\text{Loss} = \tfrac{1}{2}(\mu/r_0^3 - \omega_{\text{opt}}^2)[(t_B^3/12)A]$$

or

$$\text{Loss} = (n_0^2 - \omega_{\text{opt}}^2)\frac{(\Delta v)^3}{24A^2} \tag{2}$$

It is apparent that the loss grows very rapidly as A decreases, going as $1/A^2$.

Optimal thrust-vector turn rates for circle-to-circle transfer are given in Ref. 1. At perigee:

$$\omega_{\text{opt}} = \nu_P = \frac{n}{4}\sqrt{\frac{1+e}{1-e}}\left|\frac{3-e}{1-e}\cos\beta_1 + \sqrt{1 - \left(\frac{1+e}{1-e}\sin\beta_1\right)^2}\right| \tag{3}$$

At apogee:

$$\omega_{\text{opt}} = \nu_A = \frac{(3+e)(1-e)}{(1+e)^2}\nu_P - \frac{2n\cos\beta_1}{(1+e)(1-e^2)^{1/2}} \tag{4}$$

Here, e and $n = \sqrt{\mu/\alpha^3}$ refer to the transfer orbit, and β_1 is an initial out-of-plane pointing angle found by iteration from

$$\sin\beta_2 = [(1+e)/(1-e)]\sin\beta_1$$

and

$$\sin(\beta_1 + i_1)/\sin\beta_1 = v_0^+/v_0^- = \sqrt{1+e}$$

$$\sin(\beta_2 - i_2)/\sin\beta_2 = v_f^-/v_f^+ = \sqrt{1-e}$$

$$i_1 + i_2 = \text{desired } i$$

It is convenient for time computations to adopt orbital systems of units. Intermediate perigee orbits and the transfer orbit are best described in units normalized to perigee conditions, for which $r_0 = v_0 = n_0 = \mu = 1$, and A is in surface g's. Note that the parameters that affect orbital period, such as the eccentricity e, are unaffected by that fraction of the ΔV used to perform plane changes. These parameters are functions of the "excess perigee velocity" w_P, which for the transfer orbit is equal to the normalized coplanar ΔV_P. The transfer parameters are

$$e = (1 + w_P^2) - 1$$

$$n = (1-e)^{3/2}$$

$$\text{Period} = 2\pi[2 - (1 + w_P^2)]^{-3/2}$$

For transfer with a single perigee burn and a single apogee burn:

$$\text{Loss} = (1/24A^2)[\Delta V_P^3(n_0^2 - \nu_P^2) + \Delta V_A^3(n_A^2 - \nu_A^2)] \tag{5}$$

where $n_A = \sqrt{\mu/r_A^3}$ is the final orbital rate and the transfer time is half the transfer orbit period (neglecting burn time):

$$t = \pi[2 - (1 + w_P)^2]^{-3/2} \tag{6}$$

Time is converted to absolute units:

$$t = t/n_0 \tag{7}$$

The loss formula for circle-to-circle transfer using N multiple perigee burns is based on N intermediate impulsive ΔV's. The magnitudes of these intermediate perigee burn ΔV_i's are chosen together so as to minimize the total perigee loss, subject to the sum of the ΔV_i being equal to the impulsive perigee ΔV:

$$\min_{\Delta V_i} \left[H = \sum_{i=1}^{N} (\Delta V_i)^3 - \lambda \left(\sum_{i=1}^{N} \Delta V_i - \Delta V_P \right) \right]$$

For $i = 1$ to N:

$$\frac{\partial H}{\partial (\Delta V_i)} = 3\Delta V_i^2 - \lambda = 0$$

Hence the burns are equal:

$$\Delta V_i = \sqrt{\frac{\lambda}{3}} = \frac{\Delta V_P}{N} \tag{8}$$

Likewise, for transfers using multiple apogee burns,

$$\Delta V_j = \frac{\Delta V_A}{M} \tag{9}$$

Loss for multi-burn fixed-acceleration transfers, from Eqs. (2), (8), and (9), is given:

$$\text{Loss} = (1/24A^2)(L_P/N^2 + L_A/M^2) \tag{10}$$

where:

$$L_P = \Delta V_P^3 (n_0^2 - v_P^2)$$

$$L_A = \Delta V_A^3 (n_A^2 - v_A^2)$$

Trip time for the multiple-burn transfers, again neglecting burn time, is approximated as the sum of the periods of the N-1 intermediate perigee orbits and half the period of the transfer orbit, plus the periods of the M-1 intermediate apogee orbits. The time from initiation to the first apogee burn is thus (in orbital units)

$$t_P = \sum_{i=1}^{N-1} 2\pi \left[2 - \left(1 + i\frac{w_P}{N} \right)^2 \right]^{-3/2} + \pi [2 - (1 + w_P)^2]^{-3/2} \tag{11}$$

The excess perigee velocity of the ith intermediate orbit is

$$w_i = i\frac{w_P}{N} = \left(\frac{1}{V_0} \sum_{k=1}^{i} \Delta V_k \right)_{\text{coplanar}}$$

The time from the first apogee burn to final shutdown is conveniently expressed in different orbital units, normalized to apogee conditions ($r_F = v_F = n_F = \mu = 1$). Time is then expressed in terms of w_A:

$$t'_A = \sum_{j=1}^{M-1} 2\pi \left[2 - \left(1 + j\frac{w_A}{M} \right)^2 \right]^{-3/2} \tag{12}$$

The "apogee velocity deficiency" w_j of the jth intermediate apogee burn is

$$w_j = j\frac{w_A}{M} = \left(\frac{1}{V_F} \sum_{k=1}^{j} \Delta V_k \right)_{\text{coplanar}}$$

Total time is absolute units is

$$t = t_P/n_0 + t'_A/n_A \tag{13}$$

Equations (10) and (13) now form an approximate model of the exact fixed-acceleration transfer solution, giving loss and trip time based on acceleration and numbers of burns N and M. Given a time limit and acceleration level, an acceleration level and a desired gravity loss, or a time limit and a desired loss, N and M can be iterated to find the best N-M combination. The exact program can then be run using the same parameters to give precise time and loss figures and the optimal thrust direction history. The optimality of N and M can also be checked by comparing exact results for neighboring points.

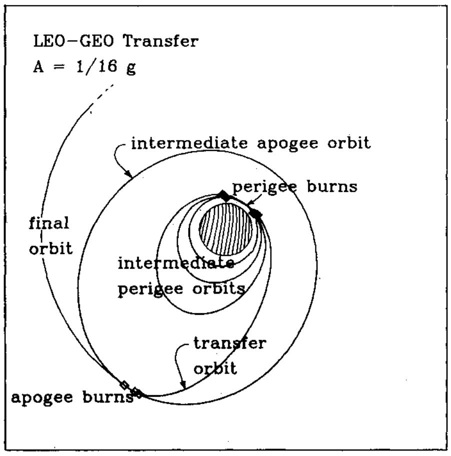

Fig. 1 Example transfer with $N = 4$, $M = 2$.

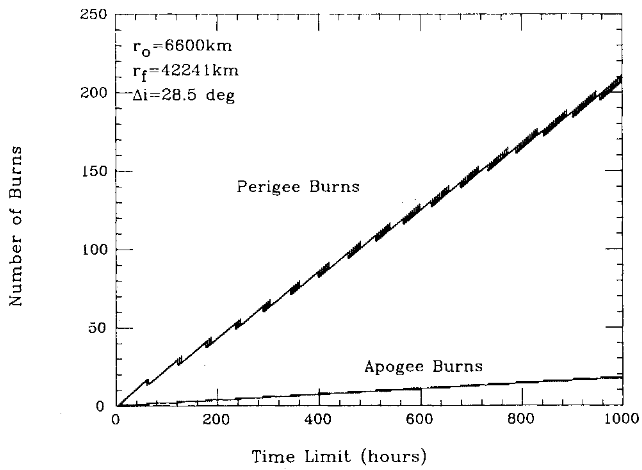

Fig. 2 Number of burns vs trip time for fixed-acceleration transfers.

Exact and approximate results computed in this fashion for the case of LEO-GEO transfer are presented in Figs. 2 and 3, for trip times of up to 1000 h and acceleration as low as (1/1000) g, for transfers with a loss of less than 1000 ft/s. Figure 2 presents the optimal numbers of burns vs trip time, showing N and M to increase roughly linearly with t, with N greater than M, as expected. Note that an increase in M results in a decrease of 3 in N, which is a result of the greater time required to complete an intermediate apogee orbit. Chatter between different numbers of burns occurs in regions where the loss for a particular M is very close to the loss for the next larger M. The advantage changes with each increase in N (which occur at different times) until the larger-M transfers become distinctly better.

Figure 3 presents the optimal loss vs trip time and acceleration level, showing that the loss decreases roughly as $1/t^2$, for a given acceleration level. Comparison between the exact and approximate results shows a very close match for low loss transfers. As the loss increases, the approximate results become less accurate: For the same N and M, they are about 5% high at 500 ft/s loss and about 10% high at 800 ft/s loss. Time estimates, for fairly large t, are consistently accurate to within 1%.

The simplicity of the approximate model allows an additional, closed-form analysis to verify the trends revealed in Figs. 2 and 3 for very long duration transfers. If t is so large that M as well as N are considerably larger than unity, the sum in Eq. (11) can be approximated as an integral:

$$t_P = 2\pi \int_1^N \left[2 - \left(1 + \frac{iw_P}{N}\right)^2\right]^{-3/2} \mathrm{d}i - \pi[2-(1+w_P)^2]^{-3/2}$$

$$t_P = \frac{2\pi N}{w_P}\left(\frac{w_P+1}{\sqrt{1-2w_P-w_P^2}} - 1\right) - \frac{\pi}{(1-2w_P-w_P^2)^{3/2}}$$

Likewise, for Eq. (13),

$$t_A = \frac{2\pi M}{w_A}\left(\frac{w_A-1}{\sqrt{1+2w_A-w_A^2}} + 1\right) - \frac{\pi}{(1+2w_A-w_A^2)^{3/2}}$$

for LEO-GEO transfers, these expressions give

$$t_P = 3.60N - 5.28 \text{ h}$$

$$t_A = 15.1M - 5.28 \text{ h}$$

$$N = 0.278t_P + 1.47$$

$$M = 0.0662t_A + 0.350$$

Writing the loss in terms of t_P and some fixed time limit $t_l = t_P + t_A$:

$$\text{Loss} = \frac{1}{A^2}\left[\frac{L_P}{(0.278t_P+1.47)^2} + \frac{L_A}{[0.0662(t_l - t_P)+0.350)]^2}\right]$$

Minimizing the loss with respect to t_P now gives

$$t_P = 0.728t_l + 2.41 \text{ h}$$

$$t_A = 0.272t_l - 2.41 \text{ h}$$

$$N = 0.2025t_l + 2.14$$

$$M = 0.0180t_l + 0.190$$

$$\text{Loss} = (1/A^2)\,[362/(t_l + 10.55)^2]$$

Table 1 presents a comparison between the exact solution, the approximate model of Eqs. (10) and (13), and the closed-form approximations for a representative transfer.

Approximate Solution for Fixed-Thrust Transfers

Robbins's formula for the gravity loss of a single fixed-thrust burn maneuver is

$$\text{Loss} = \frac{1}{(T/m_0)^2}\left[(n_0^2 - w_{\text{opt}}^2)\frac{c^3}{2}\left(1 - \frac{m}{m_0}\right)^2\left(\frac{1}{1-m/m_0} - \frac{1}{2} + \frac{1}{\ln(m/m_0)}\right)\right] \quad (14)$$

where T/m_0 is the initial thrust-to-mass ratio (in g's), and c is the characteristic exhaust velocity ($c = I_{sp} \cdot g_0$). As in the fixed-acceleration case, the loss is nearly proportional to $1/r_0^3$ and to the inverse of the square of the initial acceleration; the dependence on the burn ΔV is more complex.

Keeping in mind the changes in spacecraft mass due to thrusting, the loss for multiple perigee and apogee burn circle-to-circle transfer can be written in terms of the burn mass ratios as

$$\text{Loss} = \frac{1}{(T/m_0)^2}\left[\sum_{i=1}^{N} L_P\left(\frac{m_{i-1}}{m_0}\right)^2 + \sum_{j=1}^{M} L_A\left(\frac{m_{j-1}}{m_0}\right)^2\right]$$

$$L_P = (n_0^2 - v_P^2)\frac{c^3}{2}\left(1 - \frac{m_i}{m_{i-1}}\right)^2\left(\frac{1}{1-m_i/m_{i-1}} - \frac{1}{2} + \frac{1}{\ln(m_i/m_{i-1})}\right)$$

$$L_A = (n_A^2 - v_A^2)\frac{c^3}{2}\left(1 - \frac{m_j}{m_{j-1}}\right)^2\left(\frac{1}{1-m_j/m_{j-1}} - \frac{1}{2} + \frac{1}{\ln(m_j/m_{j-1})}\right)$$

Here the ν's are the thrust vector turn rates given in Eqs. (3) and (4).

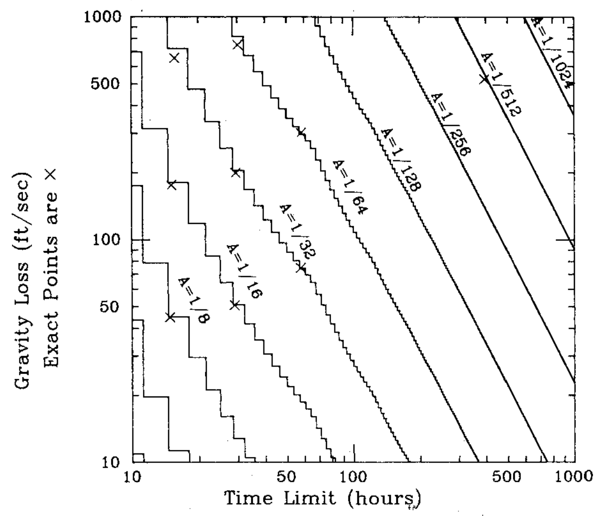

Fig. 3 Gravity loss vs trip time and acceleration for fixed-acceleration LEO-GEO transfers.

Table 1 Optimal loss and number of burns for LEO-GEO transfer under 400-h 28.5-deg plane change; $A = (1/512)g$

	Closed-form	Model	Exact
N	83.1	82	83
M	7.4	8	8
Loss, ft/s	563	545	516
Time, h	400	397.3	398

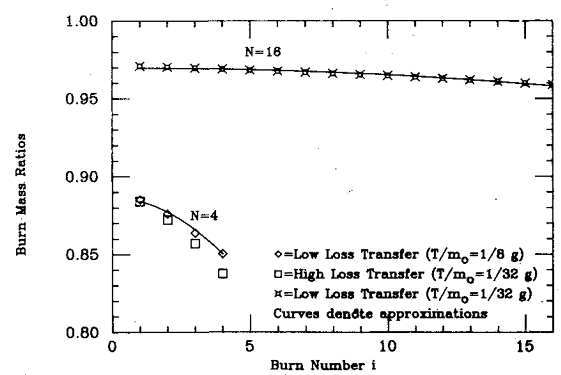

Fig. 4 Exact and approximate mass ratios: fixed-thrust LEO-GEO transfers.

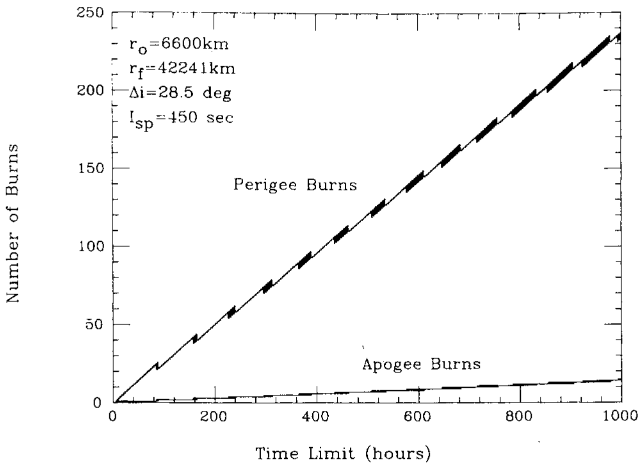

Fig. 5 Number of burns vs trip time for fixed-thrust transfers.

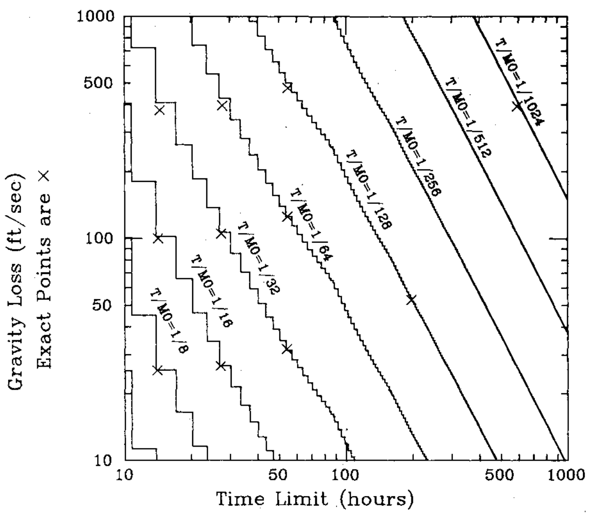

Fig. 6 Gravity loss vs trip time and acceleration for fixed-thrust LEO-GEO transfers.

As for the fixed-acceleration case, the N perigee burns and M apogee burns are chosen to minimize the transfer loss while constraining the sum of the perigee burn ΔV's to be equal to the impulsive perigee burn ΔV, and the sum of the apogee burn ΔV's to be equal to the impulsive apogee burn ΔV. Applying the rocket law, these constraints are equivalent to requiring that the products of the N perigee burn mass ratios be equal to the impulsive case perigee burn mass ratio; similarly for the M apogee burn mass ratios.

Writing R_i for the ith mass ratio m_i/m_{i-1} and R_P for the total perigee burn mass ratio, the problem for the perigee mass ratios is stated:

$$\min_{R_i} \left\{ H = \sum_{i=1}^{N} \left[\prod_{k=1}^{i-1} (R_k)^2 (1-R_i)^2 \left(\frac{1}{1-R_i} - \frac{1}{2} + \frac{1}{\ln R_i} \right) \right] - \lambda \left(\prod_{i=1}^{N} R_i - R_P \right) \right\}$$

$$\prod_{i=1}^{N} R_i = R_P = \exp\left(-\frac{\Delta V_P}{c/g} \right)$$

The optimality conditions $dH/dR_i = 0$ and the constraint give a nonlinear system of $N+1$ equations in the $N+1$ unknowns R_i and λ, which can be solved for the R_i using a numerical method.

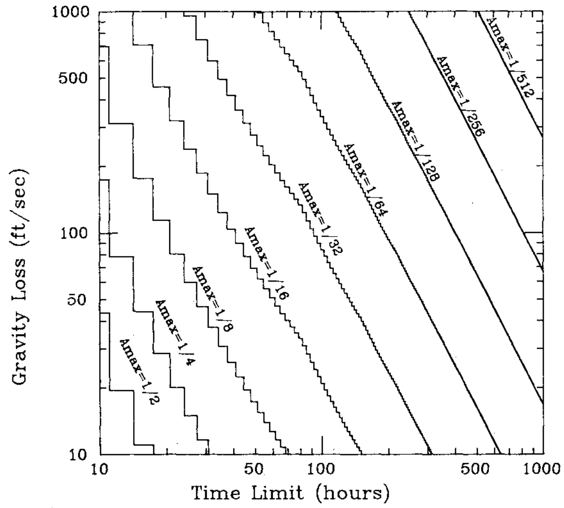

Fig. 7 Gravity loss vs trip time and maximum acceleration for fixed-thrust LEO-GEO transfers.

A much simpler and faster approach was tried with good success: A two-parameter empirical model of the mass ratios was obtained by fitting a simple function to results obtained with the exact program. Use of this model reduces the problem to the numerical solution of a single transcendental equation. Specifically, the mass ratios were found to behave nearly as $R_i = R_0 \exp(-\epsilon i^2)$, as shown on Fig. 4.

Now we can write products of functions of R_i as sums of exponents:

$$\prod_{i=1}^{N} R_i = R_0^N \exp\left(-\sum_{i=1}^{N} \epsilon i^2 \right) = R_0^N \exp\left(-\frac{\epsilon}{6} (2N^3 + 3N^2 + N) \right)$$

$$\prod_{k=1}^{i-1} R_k^2 = R_0^{2(i-1)} \exp\left(-\frac{\epsilon}{3} (2i^3 - 3i^2 + i) \right)$$

Loss is written in terms of the parameters R_0 and ϵ:

$$\text{Loss}_P = \frac{(n_0^2 - v_P^2) c^3}{2} \sum_{i=1}^{N} \left\{ R_0^{2(i-1)} \exp\left(-\frac{\epsilon}{3} (2i^3 - 3i^2 + i) \right) \times \left[[1 - R_0 \exp(-\epsilon i^2)] + [1 - R_0 \exp(-\epsilon i^2)]^2 \times \left(\frac{1}{\ln R_0 - \epsilon i^2} - \frac{1}{2} \right) \right] \right\}$$

The optimal R_i and loss are now found by using the constraint to eliminate R_0 from the loss expression, and then minimizing the loss with respect to ϵ to find the optimal ϵ and hence R_i and the loss. The perigee loss in terms of ϵ only is

$$\text{Loss}_P = \frac{(n_0^2 - v_P^2) c^3}{2} \sum_{i=1}^{N} [A_i (B_i + B_i^2 C_i)] \tag{15}$$

where

$$A_i = R_P^{2(i-1)/N} \exp[2\epsilon (i-1) D - \epsilon E_i]$$

$$B_i = 1 - R_P^{1/N} \exp[\epsilon (D - i^2)]$$

$$C_i = \frac{1}{(1/N) \ln R_P + \epsilon (D - i^2)} - \frac{1}{2}$$

$$D = \frac{1}{6} (2N^2 + 3N + 1)$$

$$E_i = \frac{1}{3} (2i^3 - 3i^2 + 2)$$

The optimal ϵ is found iteratively from

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}(\mathrm{Loss}_P)=0 \tag{16}$$

The constraint gives R_0 and so the R_i:

$$R_0=R_P^{1/N}\exp[(\epsilon/6)(2N^2+3N+1)],\qquad R_i=R_0\exp(-\epsilon i^2).$$

Likewise, for the loss due to the apogee burns,

$$\mathrm{Loss}_A=\frac{(n_A^2-v_A^2)c^3}{2}\sum_{j=1}^{M}[A_j(B_j+B_j^2C_j)]$$

$$R_j=R_A\exp(-\epsilon j^2) \tag{17}$$

with ϵ being determined from $\mathrm{d}/\mathrm{d}\epsilon(\mathrm{Loss}_A)=0$, as above. The total transfer loss:

$$\mathrm{Loss}=\frac{1}{(T/m_0)^2}(\mathrm{Loss}_P+\mathrm{Loss}_A)$$

Trip time is again approximated as the sum of the periods of the $N-1$ intermediate perigee orbits and half the period of the transfer orbit, plus the periods of the $M-1$ intermediate apogee orbits. The normalized ΔV's are:

$$W_i=\frac{1}{V_0}\sum_{k=1}^{i}\Delta V_k=-c\ell n\left(\prod_{k=1}^{i}R_i\right)$$

$$W_i=c[(\epsilon/6)(2i^3+3i^2+i)-i\ell nR_P]$$

The plane-change ΔV is removed and the w_i are converted to orbital units ($r_0=v_0=n_0=\mu=1$):

$$w_i=\frac{(\Delta V_P)_{\mathrm{coplanar}}}{(V_0W_p)}W_i$$

$$t_P=\sum_{i=1}^{N-1}2\pi[2-(1+w_i)^2]^{-3/2}+\pi[2-(1+w_P)^2]^{-3/2} \tag{18}$$

Similarly, for the intermediate apogee orbits,

$$\Delta V_j=c[(\epsilon/6)(2j^3+3j^2+j)-j\ell nR_A]$$

Converting to orbital units ($r_F=v_F=n_F=\mu=1$),

$$w_j=\frac{(\Delta V_A)_{\mathrm{coplanar}}}{V_FW_A}W_j,\qquad t_A'=\sum_{j=1}^{M-1}2\pi[2-(1+w_j)^2]^{-3/2} \tag{19}$$

Converting the times to absolute units,

$$t=t_P/n_0+(t_A')/n_F \tag{20}$$

As in the fixed-acceleration case, these equations form an approximate model of the exact solution that is useful in searching for optimal N and M, for given T/m_0, I_{sp} and t_l. By increasing t_l incrementally and finding new optimal N, M, and loss for the new t_l, optimal transfer solutions for large ranges of time and thrust limits were developed; these are summarized on Figs. 5 and 6.

The optimal numbers of burns N and M are shown vs time limit t_l on Figs. 5, for LEO-GEO transfer with $I_{\mathrm{sp}}=450$ s, as

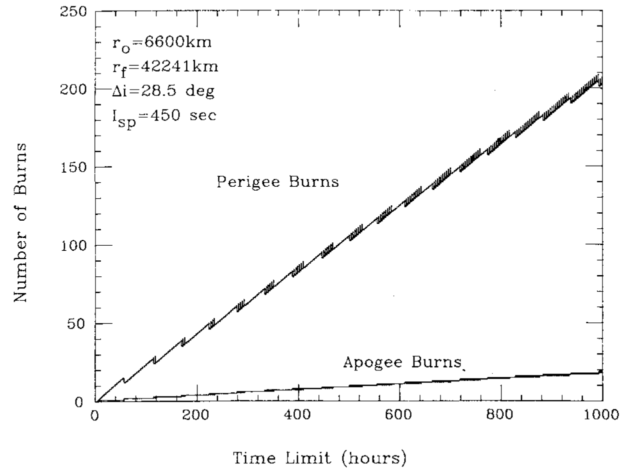

Fig. 8 Number of burns vs trip time: discrete throttling.

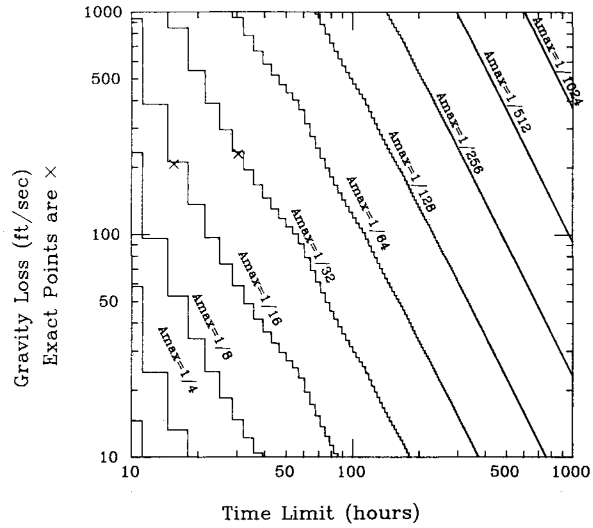

Fig. 9 Gravity loss vs trip time and maximum acceleration: discrete throttling.

computed using the approximations. Here again, N and M increase roughly linearly with t_l, with N increasing faster than M. Compared with the constant acceleration case, the fixed-thrust transfers have more perigee burns and fewer apogee burns for a given t_l, due to their lighter mass and consequent greater acceleration at apogee.

Optimal loss is shown vs trip time and initial thrust-to-mass on Fig. 6, including exact and approximate results. Here again the loss behaves nearly as $1/t^2$ for constant T/m_0. As in the fixed-acceleration case, the approximations are very good when the loss is low, but give increasing inaccurate (high) results as the loss increases. Perhaps more meaningfully, Fig. 7 shows optimal loss vs trip time and $A_{\max}$ (final thrust-to-mass); this chart provides a direct comparison with the results of the other sections.

Approximate Solution for Discretely Throttled Transfers

Here the thrust is adjusted between each burn so that maximum acceleration for the next burn (the final thrust-to-mass ratio) equals a specified acceleration limit $A_{\max}$. The derivation of the loss model proceeds as in the previous sections. The intermediate mass ratios R_i are found to be equal. The loss is written in terms of the impulsive perigee and apogee mass ratios as

$$\text{Loss} = \frac{1}{A_{\max}^2}\left| \frac{(n_0^2 - v_P^2)c^3}{2} N \frac{1 - R_P^{1/N}}{R_P^{2/N}} \left(\frac{1}{1 - R_P^{1/N}} - \frac{1}{2} + \frac{N}{\ln R_P} \right) + \frac{(n_A^2 - v_A^2)c^3}{2} M \frac{1 - R_A^{1/M}}{R_A^{2/M}} \left(\frac{1}{1 - R_A^{1/M}} - \frac{1}{2} + \frac{M}{\ln R_A} \right) \right| \tag{21}$$

The equal burns allow use of the trip time approximations developed for the fixed-acceleration case: Eqs. (11-13).

Optimal numbers of burns and optimal loss were computed as in the previous sections, with results shown on Figs. 8 and 9, respectively. N and M increase roughly as t, and the loss decreases roughly as $1/t^2$, as previously. Comparison of the approximate and exact results for loss again show a good match.

Magnitude of the loss for short t_f is quite close to the values for fixed-thrust transfers with $T/m_F = A_{\max} = (m_0/m_F)(T/m_0)$. As t_f increases, the ΔV for each burn decreases, hence the mass and acceleration are nearly constant, and the discrete throttling results approach the fixed-acceleration results.

Conclusion

This paper extends the results of a previous paper by the current author and J.V. Breakwell[1] on optimal low-thrust orbit transfers, in which the exact solution is discussed in detail. The analysis presented here also depends on a formula for gravity loss published by Robbins in 1966.[2]

Using Robbins's formula, useful approximate models for the gravity loss and trip time for fuel optimal orbit transfer are developed. Three types of propulsion system are considered. Comparison with exact results show these models to be good provided total loss is sufficiently low. Several simple trends for transfer gravity loss and trip time are revealed.

Multiple-burn transfer to geosynchronous orbit is examined in detail. The models are used to find the best number of perigee and apogee burns with respect to trip time. It is shown that for long trip times, loss varies inversely with the square of the maximum acceleration level for fixed time; and loss varies as the inverse square of the trip time for a fixed maximum acceleration level.

References

[1] Redding, D.C. and Breakwell, J.W., "Optimal Low Thrust Transfers to Synchronous Orbit," *Journal of Guidance, Control, and Dynamics,* Vol. 7, March-April 1984, pp. 148-155.

[2] Robbins, H.M., "An Analytical Study of the Impulsive Approximation," *AIAA Journal,* Vol. 4, August 1966, pp. 1417-1423.